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# Supergravity duals for $N = 2$ gauge theories

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Abstract: We construct supergravity solutions for D$p$-branes at orbifold points. The solutions are written in terms of a single function, which is the solution to a nonlinear differential equation. The near horizon limits of these solutions are dual, in the AdS/CFT sense, to super-Yang-Mills theories with 8 supercharges in various dimensions. In particular, we present a dual to $\mathcal{N} = 2$ SU($N$) SYM theory in 3+1 dimensions, and analyse some aspects of the duality.

Keywords: D-branes, Black Holes in String Theory, AdS-CFT and dS-CFT Correspondence.



 

## Contents



## 1. Introduction

The AdS/CFT correspondence [1], which provides a supergravity dual for $\mathcal{N} = 4$ super-Yang-Mills theories, can be generalized to generate supergravity duals for theories with less supersymmetry. In particular, there has been much interesting recent work on duals to $\mathcal{N} = 1$ gauge theories (including $\mathcal{N} = 1$ SU($N$) super-Yang-Mills theory) [2]–[7]. However, $\mathcal{N} = 2$ theories have received less attention.

In this paper, we will construct supergravity duals to SU($N$) super Yang-Mills theories with 8 supercharges in various dimensions. We shall do this by constructing supegravity solutions corresponding to branes on orbifolds. These solutions are complicated; nevertheless, they can be found by the methods discussed in [9]. These solutions are determined by one function, which satisfies a nonlinear differential equation. We have not found an explicit solution to this differential equation, nevertheless, the solution is determined in principle.

Our approach follows the reasoning of Maldacena which led to the AdS/CFT correspondence. In this reasoning, one notes that the theory on the worldvolume of D3-branes is a supersymmetric Yang-Mills theory. The supergravity solution corresponding to these branes is also well-known. Maldacena argued that taking the near horizon limit of the supergrav-



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ity solution was precisely the limit where the field theory degrees of freedom were isolated. Accordingly, string theory in the near horizon limit of the supergravity solution for the D3-branes (here $AdS_5 \times S^5$) was conjectured to be dual to the field theory on the worldvolume of D3-branes (in this case, the theory is SU($N$) Yang-Mills theory with 16 supercharges.)

To apply this reasoning to $\mathcal{N} = 2$ theories, we need a system of branes such that their worldvolume theory is a gauge theory with 8 supercharges. Such a system is provided by D-branes on orbifolds. In particular, the prescription of Douglas and Moore [10] (see also [11]) tells us that $N_5$ D5-branes wrapped on a 2-cycle of a $T^4/\mathbb{Z}_N$ orbifold have a worldvolume theory which is 4-dimensional SU($N_5$) super Yang-Mills theory with 8 supercharges. The theory also contains a hypermultiplet in the adjoint representation.

We can now follow the general reasoning of Maldacena. We can find the supergravity solution produced by these D5-branes. The near horizon limit of this geometry is then dual to the worldvolume theory, i.e. it is dual to 4-dimensional $\mathcal{N} = 2$ super Yang-Mills theory with a hypermultiplet.

Before embarking on details, we will compare this to other approaches which also provide theories with 8 supercharges. In the work of [12], supergravity solutions corresponding to D4-branes ending on NS5-branes were found. In such a situation, the low energy theory on the D4-branes is also 4-dimensional $\mathcal{N} = 2$ super Yang-Mills theory. Yet other constructions were given in [14].

These approaches do provide gauge theories with 8 supercharges in the low energy limit. But at high energies they differ from gauge theories (hence the full theory is not exactly $\mathcal{N} = 2$ Yang-Mills theory.) In some cases, the theory is regulated in the UV by a 6-dimensional conformal field theory, and in other cases, it is regulated by a theory with a larger gauge group in 4-dimensions. Furthermore, some of these theories live on $S^3 \times \mathbb{R}$ rather than $\mathbb{R}^{3,1}$.

The construction we present is dual to exactly $\mathcal{N} = 2$ super-Yang-Mills theory on $\mathbb{R}^{3,1}$ with gauge group SU($N$). This is why it differs from the other constructions. The existence of a hypermultiplet also leads to a difference. (Although, as we shall see, even the construction we present requires a particular limit to get the required field theory.)

## 2. Orbifolds

We shall start by considering a D2-brane on a $T^4/\mathbb{Z}_N$ orbifold. It will prove extremely helpful to consider not the singular limit, but a (partially) resolved orbifold where one two-cycle has been resolved. We shall look at a D2-brane wrapped on this resolved 2-cycle.

The metric for the resolved orbifold is known exactly to be

$$ds^2 = H^{-1}(dx_9 + A_7 dx^7 + A_8 dx^8)^2 + H(dx_7^2 + dx_8^2 + dx_6^2) \tag{2.1}$$

with

$$\begin{aligned} \partial_6 A_8 &= \partial_7 H \qquad \partial_6 A_7 = -\partial_8 H \\ \partial_7 A_8 - \partial_8 A_7 &= -\partial_6 H \,. \end{aligned}$$

There is a periodic identification of $x_9$ with period $4\pi$.

(We have chosen an unconventional gauge choice where $A_6 = 0$. Retaining $A_6$ leads to a more symmetric set of equations: see, for example [11].)

Here $H$ is a harmonic function satisfying $(\partial_6^2 + \partial_7^2 + \partial_8^2)H = 0$. If we want to resolve one 2-cycle in a $\mathbb{Z}_N$ orbifold, we take

$$H = \frac{N_1}{x_6^2 + x_7^2 + x_8^2} + \frac{N_2}{(x_6 - a)^2 + x_7^2 + x_8^2} \tag{2.2}$$

with $N_1 + N_2 = N$. $a$ is a parameter measuring the size of the resolved two-cycle.

The metric for the resolved orbifold is clearly very similar to the metric of parallel 5-branes. In the particular case above, we can think of this metric as describing two sets of 5-branes separated along the $x_6$ direction by a distance $a$. Following this intuition, we can consider a 2-brane that extends between the 5-branes at $x_6 = 0$ and $x_6 = a$, where the second worldvolume direction of the 2-brane wraps the compact $x_9$ direction. In the orbifold language, this is a 2-brane wrapped on the resolved two-cycle.

The supergravity solution for these D2-branes can be found using the methods described in [9] for the construction of intersecting brane solutions. We shall describe this construction in great detail in the next section.

However, the orbifold point also has a nonzero B-field turned on [13]. This B-field induces a D0-brane charge on the D2-brane. We should therefore look for a solution with both D2-brane and D0-brane charge. This can be done by lifting the D2-brane solution to 11 dimensions, and boosting it, thereby adding D0-brane charge. By an appropriate choice of the boost parameter, we can tune the B-field on the 2-cycle to any value desired.

The solutions for the other branes on the orbifold can be found by T-duality. In particular, T-dualizing thrice, we get a D5-brane wrapped on the orbifold 2-cycle, which as we have seen, is dual to 3+1 dimensional $\mathcal{N} = 2$ Yang-Mills theory.

We will eventually take the parameter $a$ to zero. It is nevertheless important to keep $a$ nonzero during the calculation and only take it to zero later. This will be seen explicitly in the solution where we will find a parameter $\beta$ that diverges if we set $a$ to zero immediately.

The physical meaning of this divergence is very clear. When we take $a$ to zero, the size of the two cycle shrinks to zero. The 2-brane wrapped on this 2-cycle shrinks to a point. If we start from a D2-D0 brane system and immediately set $a$ to zero, the final result is naively that of a D0-brane at the singular point. This is incorrect; we must have a nonzero D2-brane charge. To keep track of the presence of the 2-brane, it is therefore necessary to keep the parameter $a$ nonzero during the calculation, and later take the limit $a$ going to zero, keeping the D2-brane charge fixed.

A nonzero value of $a$ corresponds in the gauge theory to turning on Fayet-Iliopoulos terms [10]. The singular behaviour as $a$ goes to zero is then mapped to the singularity in the gauge theory when the Fayet-Iliopoulos terms vanish; in gauge theory terms this is due to the opening up of the Coulomb branch.



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## 3. 2-brane on $\mathbb{Z}_N$ orbifold

### 3.1 Notation

We will start by constructing the supergravity solution for a D2-brane wrapped on a 2-cycle of a $\mathbb{Z}_N$ orbifold. We can approach the supergravity solution of this system in the same way as [9]. First we establish some notation.

The intuition which may be helpful here is to regard the orbifold metric as being the metric of two sets of 5-branes oriented along $x_0$, $x_1$, $x_2$, $x_3$, $x_4$, $x_5$. The first set of $N_1$ 5-branes is at $x_6 = 0$, and the second set of $N_2$ 5-branes is at $x_6 = a$. The two-brane is oriented along $x_0$, $x_1$, $x_6$ and stretches between the two sets of 5-branes.

This state is supersymmetric. This implies that there must exist spinors satisfying the Killing spinor equations. We will write down these equations below. We shall then find an ansatz for the Killing spinor, and use this ansatz to constrain the solution. That is, we write down the Killing spinor first, and then find the appropriate background fields such that the spinor actually satisfies the Killing equation. The justification for this procedure is that it should be self-consistent.

We will denote the directions $x_1$, $x_2$, $x_3$, $x_4$, $x_5$ collectively by $x_a$. The directions $x_7$, $x_8$ will collectively be called $x_\alpha$.

The approach in [9] used an analysis of the condition for conserved supercharges (the Killing spinor equation)

$$\partial_\mu \epsilon + \frac{1}{2}\omega_\mu{}^{ab}\gamma_{ab}\epsilon + \left(e^\Phi G_\mu{}^{abc}\gamma_{abc} - e^\Phi G_{abcd}\gamma_\mu{}^{abcd}\right)\epsilon = 0\,, \tag{3.1}$$

where $G_{abcd}$ is the field strength for the field that couples to the 2-brane. We have normalized it so that the Killing equations take a simple form.

We will make the following ansatz (see [9] for a more detailed discussion)

- $\epsilon = (g_{00})^{1/4}\epsilon_0$ where $\epsilon_0$ is a constant spinor.
- The constant spinor satisfies

$$\left(1 - \gamma^{069}\right)\epsilon_0 = \left(1 - \gamma^{6789}\right)\epsilon_0 = 0\,. \tag{3.2}$$

- The metric components will be taken to be diagonal with the addition of $e_{9\tilde{7}}, e_{9\tilde{8}}$ (present already in (2.1)) and $e_{6\tilde{a}}$ (induced by the 2-brane).
- The nonzero gauge field strengths are

$$G_{069a} \quad G_{069\alpha} \quad G_{09a\alpha} \quad G_{078a} \quad G_{0678} \quad G_{06a\alpha}\,.$$

### 3.2 Supersymmetry equations

The Killing spinor equations (3.1) now reduce to a set of algebraic equations which determine the field strengths in terms of the spin connections and also impose some constraints on the spin connections.

To figure out these algebraic equations, note that the constraints (3.2) are singlets under rotations in $x_a, x_\alpha$. Hence we can decompose the equations in representations of the rotations in $x_a, x_\alpha$.

For instance, the field strengths and spin connections which transform under rotations in $x_1$, but are singlets under other rotations, are $\omega_A^{\ A1}, G_{0691}$, and $H_{0781}$. Hence these terms must cancel against each other in each Killing spinor equation (3.1) with any $\mu$. For $\mu = 0, 4, 6, 7, 9$, we find respectively

$$\begin{aligned}
{\omega_0}^{01}\gamma_1\epsilon_0 + e^\Phi G_{0691}\gamma^{0691}\epsilon_0 + e^\Phi G_{0781}\gamma^{0781}\epsilon_0 &= 0\\
{\omega_4}^{41}\gamma_1\epsilon_0 - e^\Phi G_{0691}\gamma^{0691}\epsilon_0 - e^\Phi G_{0781}\gamma^{0781}\epsilon_0 &= 0\\
{\omega_6}^{61}\gamma_1\epsilon_0 + e^\Phi G_{0691}\gamma^{0691}\epsilon_0 - e^\Phi G_{0781}\gamma^{0781}\epsilon_0 &= 0\\
{\omega_7}^{71}\gamma_1\epsilon_0 - e^\Phi G_{0691}\gamma^{0691}\epsilon_0 + e^\Phi G_{0781}\gamma^{0781}\epsilon_0 &= 0\\
{\omega_9}^{91}\gamma_1\epsilon_0 + e^\Phi G_{0691}\gamma^{0691}\epsilon_0 - e^\Phi G_{0781}\gamma^{0781}\epsilon_0 &= 0\,.
\end{aligned} \tag{3.3}$$

From these equations we find

$$\begin{aligned}
e^\Phi G_{0691}\gamma^{0691}\epsilon_0 &= -\frac{1}{2}\left({\omega_0}^{01} - {\omega_7}^{71}\right)\gamma_1\epsilon_0\\
e^\Phi G_{0781}\gamma^{0781}\epsilon_0 &= -\frac{1}{2}\left({\omega_0}^{01} + {\omega_7}^{71}\right)\gamma_1\epsilon_0
\end{aligned} \tag{3.4}$$

and the constraints

$$\begin{aligned}
{\omega_4}^{41} + {\omega_0}^{01} &= 0\\
{\omega_6}^{61} + {\omega_7}^{71} &= 0\\
{\omega_9}^{91} + {\omega_7}^{71} &= 0\,.
\end{aligned} \tag{3.5}$$

We can use the equation (3.2) to solve for the field strengths; we find

$$e^\Phi G_{0691} = \frac{1}{2}\left({\omega_0}^{01} - {\omega_7}^{71}\right) \qquad e^\Phi G_{0781} = -\frac{1}{2}\left({\omega_0}^{01} + {\omega_7}^{71}\right). \tag{3.6}$$

The other terms in the Killing spinor equations can be similarly analyzed to find the field strengths and other constraints on the metric. The constraints on the metric will be solved in the following subsection; here we present the field strengths in terms of the spin connections. These are found to be

$$\begin{aligned}
e^\Phi G_{078a} &= -\frac{1}{2}\left(\omega_0^{0a} + \omega_7^{7a}\right) & e^\Phi G_{068a} &= \frac{1}{2}\omega_6^{a7} & &\\
e^\Phi G_{067a} &= -\frac{1}{2}\omega_6^{a8} & e^\Phi G_{0678} &= -\omega_0^{06} & &\\
e^\Phi G_{069a} &= \frac{1}{2}\left(\omega_0^{0a} - \omega_7^{7a}\right) & e^\Phi G_{09a\alpha} &= \frac{1}{2}\omega_6^{a\alpha} & e^\Phi G_{069\alpha} &= \omega_0^{0\alpha}\,.
\end{aligned} \tag{3.7}$$



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### 3.3 The solution

As was shown in [9], we can solve the constraints on the metric by the ansatz

$$\begin{aligned} e_{0\tilde{0}} &= \lambda^{-1/4} & e_{a\tilde{a}} &= \lambda^{1/4} & e_{6\tilde{6}} &= \lambda^{-1/4}H^{1/2} \\ e_{7\tilde{7}} &= e_{8\tilde{8}} = \lambda^{1/4}H^{1/2} & e_{9\tilde{9}} &= \lambda^{-1/4}H^{-1/2} \\ e_{9\tilde{7}} &= e_{9\tilde{9}}A_7 & e_{9\tilde{8}} &= e_{9\tilde{9}}A_8 & e_{6\tilde{a}} &= e_{6\tilde{6}}\phi_a \end{aligned} \tag{3.8}$$

with the remaining constraints

$$\begin{aligned} \partial_7 A_8 - \partial_8 A_7 &= -\partial_6(H\lambda) \\ \partial_6 A_8 &= \partial_7 H \qquad \partial_6 A_7 = -\partial_8 H \\ \partial_6(\phi_a H) &= \partial_a H\,. \end{aligned} \tag{3.9}$$

Similarly by requiring the variation of the dilatino to vanish, we find that the dilaton is given by

$$e^{\Phi} = \lambda^{1/4}\,. \tag{3.10}$$

We can then show that the field strengths (3.4), (3.7) can be obtained from the gauge fields

$$\begin{aligned} C_{09a} &= -\frac{1}{4}\frac{\phi_a}{\lambda} & C_{069} &= \frac{1}{4\lambda} \\ C_{06\alpha} &= \frac{A_\alpha}{4\lambda} & C_{0a\alpha} &= \frac{A_\alpha \phi_a}{4\lambda} \\ C_{078} &= -\frac{H}{4}\,. \end{aligned} \tag{3.11}$$

Finally we impose the equations of motion for pointlike sources. These yield the equation

$$\partial_a \phi_a = H^{-1}\partial_6\lambda + \partial_6\left(\frac{\phi_a^2}{2}\right). \tag{3.12}$$

This provides a complete solution for the 2-brane on a 2-cycle of a $\mathbb{Z}_N$ orbifold.

### 3.4 Adding a B-field

In the solution of the previous section, we were at the point in moduli space where the B-field on the orbifold was zero, as could be seen from the fact that the D0-brane charge was zero. Turning on a B-field on the two-cycle will add zero-brane charge to the system.

To find a solution with added zero-brane charge, we lift the solution we have found to 11 dimensions (thus getting a M2-brane on the orbifold.) We then boost in the 11th direction and dimensionally reduce, thus obtaining a solution with D2+D0 charge.

All these steps are straightforward, and we can directly present the final answer in the next section. The solutions for other branes on the orbifold can be obtained by T-duality.

It should be noted that since the M-theory direction is compact, the boost is not a symmetry of the theory. We will therefore generate a new solution of supergravity which is physically inequivalent to the old solution. Also note that when we lift the D-brane solution to 11 dimensions, we find a M2-brane solution which is delocalized in the 11th dimension. Nevertheless, it is possible to boost this solution, and we obtain a boosted solution which is still delocalized. We therefore do not have to perform any "smearing" of the solution.

## 4. Branes on orbifolds

### 4.1 Notation

First of all we summarize the various formulae that are required in the solutions.

The solutions are expressed in terms of the functions $\lambda, H, A_7, A_8$. In addition we will introduce a constant $\beta$ and a function $\Delta$ defined as

$$\Delta = \cosh^2 \beta - \lambda^{-1} \sinh^2 \beta \,. \tag{4.1}$$

The constant $\beta$ controls the value of the B-field on the 2-cycle. $\beta = 0$ corresponds to zero B-field.

The functions satisfy the differential equations

$$\begin{aligned} \partial_7 A_8 - \partial_8 A_7 + \partial_6 (H\lambda) &= Q(x_a, x_6)\delta(x_\alpha) \\ \partial_6 A_8 &= \partial_7 H \qquad \partial_6 A_7 = -\partial_8 H \\ \partial_6(\phi_a H) &= \partial_a H \\ \partial_a \phi_a &= H^{-1}\partial_6 \lambda + \partial_6 \left(\frac{\phi_a^2}{2}\right) . \end{aligned} \tag{4.2}$$

These functions can be expressed in terms of a single function $\tau$ through

$$\begin{aligned} H\phi_a &= \partial_a \partial_6 \tau \\ H &= \partial_6^2 \tau \\ \lambda + H\phi_a^2 &= \partial_a^2 \tau \,. \end{aligned} \tag{4.3}$$

$\tau$ then satisfies the differential equation

$$\partial_a^2 \tau + \partial_6^2 \partial_a^2 \tau - (\partial_6 \partial_a \tau)^2 = \frac{1}{\partial_6} Q(x_a, x_6) \delta(x_\alpha) \,. \tag{4.4}$$

The function $Q$ parametrizes the brane source; in particular, it incorporates the effects of brane bending.

We will denote the NSNS 2-form by $B_{\mu\nu}$, and the RR forms by $C^{(k)}_{\mu_1,\ldots,\mu_k}$.

### 4.2 2-brane on $\mathbb{Z}_N$ orbifold

Here $a$ runs from 1 to 5. The metric is

$$\begin{aligned} ds^2 &= \lambda^{-1/2}\Delta^{-1/2}(-dt^2 + \Delta dx_a^2) + \lambda^{-1/2}\Delta^{1/2} H (dx^6 + \phi_a dx^a)^2 + \\ &\quad + \lambda^{1/2}\Delta^{1/2} H(dx_7^2 + dx_8^2) + \lambda^{-1/2}\Delta^{1/2} H^{-1}(dx_9 + A_7 dx^7 + A_8 dx_8)^2 \,. \end{aligned} \tag{4.5}$$

The dilaton is

$$e^{\Phi} = \lambda^{1/4}\Delta^{3/4} \,. \tag{4.6}$$

The gauge fields are

$$
\begin{aligned}
C^{(1)}_0 &= \sinh\beta\cosh\beta\frac{(1-\lambda)}{\Delta\lambda} \\
C^{(3)}_{09a} &= -\frac{1}{4}\frac{\phi_a}{\lambda}\cosh\beta & C^{(3)}_{069} &= \frac{1}{4\lambda}\cosh\beta \\
C^{(3)}_{06\alpha} &= \frac{A_\alpha}{4\lambda}\cosh\beta & C^{(3)}_{0a\alpha} &= \frac{A_\alpha\phi_a}{4\lambda}\cosh\beta \\
C^{(3)}_{078} &= -\frac{H}{4}\cosh\beta \\
B_{9a} &= -\frac{3}{8}\frac{\phi_a}{\lambda}\sinh\beta & B_{69} &= \tfrac{3}{8\lambda}\sinh\beta \\
B_{6\alpha} &= \tfrac{3A_\alpha}{8\lambda}\sinh\beta & B_{a\alpha} &= \tfrac{3A_\alpha\phi_a}{8\lambda}\sinh\beta \\
B_{78} &= -\tfrac{3H}{8}\sinh\beta\,.
\end{aligned}
\tag{4.7}
$$

### 4.3 3-brane on $\mathbb{Z}_N$ orbifold

Here $i$ runs over $0,1$, $a$ runs over $2,3,4,5$. The metric is

$$
\begin{aligned}
ds^2 =\ & \lambda^{-1/2}\Delta^{-1/2}(-dt^2+dx_1^2+\Delta dx_a^2)+\lambda^{-1/2}\Delta^{1/2}H(dx^6+\phi_a dx^a)^2+ \\
& +\lambda^{1/2}\Delta^{1/2}H(dx_7^2+dx_8^2)+\lambda^{-1/2}\Delta^{1/2}H^{-1}(dx_9+A_7dx^7+A_8dx_8)^2\,.
\end{aligned}
\tag{4.8}
$$

The dilaton is

$$
e^\Phi = \Delta^{1/2}\,. \tag{4.9}
$$

The gauge fields are

$$
\begin{aligned}
C^{(2)}_{01} &= \sinh\beta\cosh\beta\frac{(1-\lambda)}{\Delta\lambda} \\
C^{(4)}_{abc7} &= \frac{1}{4}\epsilon_{abcd}(\phi_d A_7+\partial_8\partial_d\tau) & C^{(4)}_{abc8} &= \frac{1}{4}\epsilon_{abcd}(\phi_d A_8-\partial_7\partial_d\tau) \\
C^{(4)}_{abc9} &= \frac{1}{4}\epsilon_{abcd}\phi_d \\
B_{9a} &= -\frac{3}{8}\frac{\phi_a}{\lambda}\sinh\beta & B_{69} &= \frac{3}{8\lambda}\sinh\beta \\
B_{6\alpha} &= \frac{3A_\alpha}{8\lambda}\sinh\beta & B_{a\alpha} &= \frac{3A_\alpha\phi_a}{8\lambda}\sinh\beta \\
B_{78} &= -\frac{3H}{8}\sinh\beta\,.
\end{aligned}
\tag{4.10}
$$

### 4.4 4-brane on $\mathbb{Z}_N$ orbifold

Here $i$ runs over $1,2$, $a$ runs over $3,4,5$. The metric is

$$
\begin{aligned}
ds^2 =\ & \lambda^{-1/2}\Delta^{-1/2}(-dt^2+dx_i^2+\Delta dx_a^2)+\lambda^{-1/2}\Delta^{1/2}H(dx^6+\phi_a dx^a)^2+ \\
& +\lambda^{1/2}\Delta^{1/2}H(dx_7^2+dx_8^2)+\lambda^{-1/2}\Delta^{1/2}H^{-1}(dx_9+A_7dx^7+A_8dx_8)^2\,.
\end{aligned}
\tag{4.11}
$$

The dilaton is

$$
e^\Phi = \lambda^{-1/4}\Delta^{1/4}\,. \tag{4.12}
$$

The gauge fields are

$$\begin{aligned}
C^{(3)}_{012} &= \sinh\beta\cosh\beta\frac{(1-\lambda)}{\Delta\lambda} \\
C^{(3)}_{ab7} &= \frac{1}{4}\epsilon_{abc}(\phi_c A_7 + \partial_8\partial_c\tau) \qquad & C^{(3)}_{ab8} &= \frac{1}{4}\epsilon_{abc}(\phi_c A_8 - \partial_7\partial_c\tau) \\
C^{(3)}_{ab9} &= \frac{1}{4}\epsilon_{abc}\phi_c \\
B_{9a} &= -\frac{3}{8}\frac{\phi_a}{\lambda}\sinh\beta & B_{69} &= \frac{3}{8\lambda}\sinh\beta \\
B_{6\alpha} &= \frac{3A_\alpha}{8\lambda}\sinh\beta & B_{a\alpha} &= \frac{3A_\alpha\phi_a}{8\lambda}\sinh\beta \\
B_{78} &= -\frac{3H}{8}\sinh\beta\,.
\end{aligned} \tag{4.13}$$

### 4.5 5-brane on $\mathbb{Z}_N$ orbifold

Here $i$ runs over $1,2,3$, $a$ runs over $4,5$. The metric is

$$\begin{aligned}
ds^2 = \lambda^{-1/2}\Delta^{-1/2}(-dt^2 + dx_i^2 + \Delta dx_a^2) + \lambda^{-1/2}\Delta^{1/2}H(dx^6 + \phi_a dx^a)^2 + \\
+ \lambda^{1/2}\Delta^{1/2}H(dx_7^2 + dx_8^2) + \lambda^{-1/2}\Delta^{1/2}H^{-1}(dx_9 + A_7 dx^7 + A_8 dx_8)^2\,.
\end{aligned} \tag{4.14}$$

The dilaton is

$$e^{\Phi} = \lambda^{-1/2}\,. \tag{4.15}$$

The gauge fields are

$$\begin{aligned}
C^{(4)}_{0123} &= \sinh\beta\cosh\beta\frac{(1-\lambda)}{\Delta\lambda} \\
C^{(2)}_{a7} &= \frac{1}{4}\epsilon_{ab}(\phi_b A_7 + \partial_8\partial_b\tau) \qquad & C^{(2)}_{a8} &= \frac{1}{4}\epsilon_{ab}(\phi_b A_8 - \partial_7\partial_b\tau) \\
C^{(2)}_{a9} &= \frac{1}{4}\epsilon_{ab}\phi_b \\
B_{9a} &= -\frac{3}{8}\frac{\phi_a}{\lambda}\sinh\beta & B_{69} &= \frac{3}{8\lambda}\sinh\beta \\
B_{6\alpha} &= \frac{3A_\alpha}{8\lambda}\sinh\beta & B_{a\alpha} &= \frac{3A_\alpha\phi_a}{8\lambda}\sinh\beta \\
B_{78} &= -\frac{3H}{8}\sinh\beta\,.
\end{aligned} \tag{4.16}$$

## 5. Comments on the duality

The last subsection above describes the supergravity solution for 5-branes wrapped on an orbifold 2-cycle. By taking the near-horizon limit, we obtain the dual to a $\mathcal{N}=2$ gauge theory in 3+1 dimensions. We now discuss some features of this duality.

First, we will obtain the as yet undetermined parameter $\beta$. This parameter tells us the ratio of the D2-brane charge to the D0-brane charge (or more generally, the ratio of the D$(p+2)$-brane charge to the D$p$-brane charge) and therefore measures the B-field through the 2-cycle. We will therefore determine it by requiring the particular value of the B-field such that we are at the perturbative orbifold point of string theory, so that the analysis of [10] is valid.

The orbifold point is defined by the point in the moduli space where $B = 1/N$ for a $\mathbb{Z}_N$ orbifold. At this point in moduli space, a single 2-brane wrapped on the orbifold gets an induced 0-brane charge equal to $1/N$. This can be used to fix the value of $\beta$ at the orbifold point.

The calculation is straightforward, given the supergravity solution. We find that (we set $\alpha' = 1$)

$$\sinh\beta = \frac{1}{4\pi N a}\,. \tag{5.1}$$

(Recall that $a$ is the separation between the centers in the orbifold metric (2.2).) This value of $\beta$ is the perturbative orbifold point. Other values of $\beta$ correspond to changing the B-field on the 2-cycle.

Secondly, one should also ask where the supergravity solution we have found is valid. For this, we need the string coupling to be small. Hence

$$e^{\Phi} = g_0\lambda^{-1/2} \ll 1\,. \tag{5.2}$$

This means that the supergravity solution cannot be trusted near the branes. This corresponds to the fact that the gauge theory is trivial in the infrared, and cannot therefore have a simple description in supergravity.

Also, we need the curvature to be small. We cannot find the exact regime where this applies, since we have not found an analytic solution. However, we know that this must break down far away from the branes, since asymptotically, the geometry is similar to that of a five-brane. This breakdown corresponds to the fact that the gauge theory is asymptotically free, and again there cannot be a good supergravity description.

Thirdly, in the limit where $a$ becomes zero, the orbifold metric (2.2) has a manifest $\mathrm{SO}(3) \cong \mathrm{SU}(2)$ symmetry. In the gauge theory, this is the R-symmetry of the $\mathcal{N} = 2$ theory. For nonzero $a$ it is broken to a $\mathrm{U}(1)$ symmetry; this is interpreted in the gauge theory as the explicit breaking of the R-symmetry by a Fayet-Iliopoulos term.

In this context, we should emphasize that the duality we have found holds even when $a$ is not taken to zero. In this case, the supergravity solution with a nonzero value of $a$ is dual to the field theory with a nonzero Fayet-Iliopoulos term (the precise mapping between the parameter $a$ and the Fayet-Iliopoulos term was given in [10]). The pure gauge theory is obtained in the limit where $a$ is taken to zero while imposing the relation (5.1) on $\beta$. Unfortunately, since we have not been able to solve the differential equation, we cannot take this limit explicitly.

Finally, we can see that in the weak coupling region, the gauge coupling has a logarithmic falloff with scale, as is expected from the gauge theory correspondence. This can be seen from the fact that the gauge coupling asymptotically satisfies the Laplace equation in 2 space dimensions, which has a logarithmic solution. This analysis is the same as that performed in the geometric engineering analysis of field theories [15, 16, 17].

## 6. Conclusion

In summary, we have constructed supergravity duals for $\mathcal{N} = 2$ gauge theories. This method can be extended to all theories which can be obtained on branes through geometric engineering. This is a very wide class of theories.

We have left the detailed analysis of the correspondence to future work. It would be interesting to see what information can be obtained about the gauge theories by this duality. In particular, it should be possible to rederive the results of [18, 19]. It should also be possible to analyze the behaviour of solitons in these theories by analyzing branes ending on other branes.

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## References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[2] J. Polchinski and M.J. Strassler, *The string dual of a confining four-dimensional gauge theory*, hep-th/0003136.

[3] I.R. Klebanov and A.A. Tseytlin, *Gravity duals of supersymmetric* $\mathrm{SU}(N) \times \mathrm{SU}(N+M)$ *gauge theories*, *Nucl. Phys.* **B 578** (2000) 123 [hep-th/0002159].

[4] I.R. Klebanov and M.J. Strassler, *Supergravity and a confining gauge theory: duality cascades and $\chi$SB-resolution of naked singularities*, *J. High Energy Phys.* **08** (2000) 052 [hep-th/0007191].

[5] B.S. Acharya, *On realising $N = 1$ super Yang-Mills in M-theory*, hep-th/0011089.

[6] J.M. Maldacena and C. Núñez, *Towards the large-N limit of pure $N = 1$ super Yang-Mills*, *Phys. Rev. Lett.* **86** (2001) 588 [hep-th/0008001].

[7] M. Atiyah, J.M. Maldacena and C. Vafa, *An M-theory flop as a large-N duality*, *J. Math. Phys.* **42** (2001) 3209 [hep-th/0011256].

[8] E. Witten, *Branes and the dynamics of QCD*, *Nucl. Phys.* **68** *(Proc. Suppl.)* (1998) 216.

[9] A. Rajaraman, *Supergravity solutions for localised brane intersections*, *J. High Energy Phys.* **09** (2001) 018 [hep-th/0007241].

[10] M.R. Douglas and G.W. Moore, *D-branes, quivers and ALE instantons*, hep-th/9603167.

[11] C.V. Johnson and R.C. Myers, *Aspects of type-IIB theory on ALE spaces*, *Phys. Rev.* **D 55** (1997) 6382 [hep-th/9610140].

[12] B. Brinne, A. Fayyazuddin, S. Mukhopadhyay and D.J. Smith, *Supergravity M5-branes wrapped on Riemann surfaces and their QFT duals*, *J. High Energy Phys.* **12** (2000) 013 [hep-th/0009047].

[13] P.S. Aspinwall, *Enhanced gauge symmetries and K3 surfaces*, *Phys. Lett.* **B 357** (1995) 329 [hep-th/9507012].

[14] J. Polchinski, $N = 2$ *gauge-gravity duals*, *Int. J. Mod. Phys.* **A 16** (2001) 707 [hep-th/0011193].

[15] S. Katz and C. Vafa, *Geometric engineering of* $N = 1$ *quantum field theories*, *Nucl. Phys.* **B 497** (1997) 196 [hep-th/9611090].

[16] S. Katz, A. Klemm and C. Vafa, *Geometric engineering of quantum field theories*, *Nucl. Phys.* **B 497** (1997) 173 [hep-th/9609239].

[17] S. Katz, P. Mayr and C. Vafa, *Mirror symmetry and exact solution of 4d* $N = 2$ *gauge theories, 1*, *Adv. Theor. Math. Phys.* **1** (1998) 53 [hep-th/9706110].

[18] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation and confinement in* $N = 2$ *supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 426** (1994) 19 [hep-th/9407087].

[19] M.R. Douglas and S.H. Shenker, *Dynamics of* $\mathrm{SU}(N)$ *supersymmetric gauge theory*, *Nucl. Phys.* **B 447** (1995) 271 [hep-th/9503163].